

Multi-granular Perspectives on Covering

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Abstract. Covering model provides a general framework for granular computing in that overlapping among granules are almost indispensable. For any given covering, both intersection and union of covering blocks containing an element are exploited as granules to form granular worlds at different abstraction levels, respectively, and transformations among these different granular worlds are also discussed. As an application of the presented multi-granular perspective on covering, relational interpretation and axiomization of four types of covering based rough upper approximation operators are investigated, which can be dually applied to lower ones.

Keywords: Covering, Granulation, Granule

1 Introduction

The ability to conceptualize the world at different granularities and to switch among these granularities is fundamental to human intelligence and flexibility [9][19]. To simulate such an ability of human problem solving, Granular Computing (GrC) was proposed. In fact, there have existed many concrete models of granular computing, such as computing with words [17][18], rough set theory [12], formal concept analysis [8], quotient space theory [19], etc.

Granulation [18][10], grouping some elements of the domain into granules, is a fundamental step towards granular computing. Idealized granulation should yield pairwise disjoint granules such as equivalence classes or partition blocks. However, incompleteness and impreciseness of information, as well as variant similarity of individuals, always result in overlapping among granules. Consequently, granular worlds are always formalized as a set equipped with a family of nonempty subsets whose union equals to the universe, which is precisely the covering model [10].

For any given covering, both union and intersection of covering blocks containing the given element can be viewed as induced granules, which form two new coverings representing granular worlds at different abstraction levels. Consequently, transformations among these three covering are important and interesting in that they simulate the process of switching among granularities, which are precisely our respect of discussion.

Pawlak's rough set provides a concrete framework performing granular computing based on partition or equivalence relation, however, absolute disjointness among granules restrict applications of classical rough set theory, so generalized rough set models based on covering are extensively researched, and many results such as axiomization and reduction theory are obtained (see, for example, in [20][21][22][23] for details). In these researches, union and intersection of covering blocks containing the given element were always called the friend and neighborhood, respectively, and they were exploited as basic knowledge to approximate unknown knowledge. However, transformations among induced granular worlds as well as relational interpretations of friend and neighborhood are not discussed, while the latter naturally results in axiomization of several types of covering based rough approximation operators.

The paper is organized as follows: Based on the definition of star and closure of an element, two granular worlds are induced in section 2, and transformations among them are also discussed. Section 3 investigates axiomization and relational interpretation of four types of covering rough approximation operators, and it is proved that these operators exploit induced granules, rather than given covering blocks, as basic knowledge to approximate unknown knowledge. Conclusions and discussions are grouped in the final section.

Throughout this paper, we do not restrict the universe of discourse U to be finite. The class of all subsets of U will be denoted by $\wp(U)$. For any $X \subseteq U$, $-X$ will be used to denote the complement of X . For any subset system $\mathcal{F} \subseteq \wp(U)$, we will denote $\{-F \mid F \in \mathcal{F}\}$ as \mathcal{F}^c . Obviously, $(\mathcal{F}^c)^c = \mathcal{F}$. Moreover, I always denotes arbitrary index set.

2 Granular Worlds Induced by Covering

In this section, we firstly recall the star and point closure system of any given covering, and then we discuss transformations among these three subset systems. When coverings are understood as granular worlds at some abstraction levels, the transformations are precisely the processes of switching among different granularities.

2.1 Star and Point Closure System of Covering

A family β of nonempty subsets of U is called a covering of U if $\bigcup\{B \mid B \in \beta\} = U$, and any $B \in \beta$ is called a covering block. Obviously, covering generalizes partition in that overlapping among covering blocks are permitted. For any $x \in U$, let β_x denote the collection of covering blocks containing x , namely $\beta_x = \{B \mid x \in B \in \beta\}$.

Definition 1 ([3][6]). *For any $x \in U$, the star and point closure of x with respect to β , denoted by $\text{star}(x, \beta)$ and $\downarrow x$, are defined as the union and intersection of covering blocks containing x , respectively.*

Formally, $\forall x \in U$, $star(x, \beta) = \bigcup \{B \mid B \in \beta_x\}$ and $\downarrow x = \bigcap \{B \mid B \in \beta_x\}$. In [22][20], $star(x, \beta)$ and $\downarrow x$ are called friend and neighborhood of x , and denoted by $Friends(x)$ and $Neighbor(x)$, respectively. In the followings, the collection of stars and point closures with respect to covering β are denoted by $\mathcal{S}(\beta)$ and $\mathcal{P}(\beta)$, namely $\mathcal{S}(\beta) = \{star(x, \beta) \mid x \in U\}$ and $\mathcal{P}(\beta) = \{\downarrow x \mid x \in U\}$. Both the star system and the point closure system are coverings of U since $x \in star(x, \beta)$ and $x \in \downarrow x$ hold for any $x \in U$.

Definition 2 ([3]). For any coverings α and β , if for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subseteq B$, then α is called a refinement of β , denoted by $\beta \sqsubseteq \alpha$.

Proposition 1. $\mathcal{S}(\beta) \sqsubseteq \beta \sqsubseteq \mathcal{P}(\beta)$.

For any $x \in U$, $star(x, \beta)$ and $\downarrow x$ are regarded as granules of different grain size, then Proposition 1 implies that the two induced coverings represent granular worlds at different abstraction levels.

Definition 3 ([6]). For any covering β , the specialization preordering \preceq_β induced by β is defined as: $\forall x, y \in U$, $y \preceq_\beta x \Leftrightarrow (\forall K \in \beta)(x \in K \rightarrow y \in K)$.

In general, specialization preordering \preceq_β is reflexive and transitive but not necessarily antisymmetric. Moreover, point closure $\downarrow x$ is precisely the principal down-set $\{y \mid y \preceq_\beta x\}$ generated by x with respect to \preceq_β . The following Proposition 2 characterizes the structure of $\mathcal{P}(\beta)$.

Proposition 2. For any covering β of U , $\forall x, y \in U$, then

1. $y \in \downarrow x$ if and only if $\downarrow y \subseteq \downarrow x$.
2. $\downarrow x = \bigcup \{\downarrow z \mid z \in \downarrow x\}$, $\downarrow x = \bigcap \{\downarrow z \mid x \in \downarrow z\}$.

For further discussion, we need to recall some notations about binary relation. For any binary relation R on U and $x \in U$, $R(x) = \{y \in U \mid xRy\}$ and $R^{-1} = \{\langle v, u \rangle \mid uRv\}$. Particularly, when \leq is a partial ordering or preordering, \leq^{-1} is always called the dual of \leq and denoted by \geq . Moreover, $\uparrow x = \{y \in U \mid x \leq y\}$ and $\downarrow x = \{y \in U \mid y \leq x\}$ are called principal up-set and principal down-set generated by x , respectively [6].

For any covering β , the subset system $\beta^c = \{-B \mid B \in \beta\}$ is not necessarily a covering of U any more. In fact, β^c is also a covering of U if and only if $\bigcap \{B \mid B \in \beta\} = \emptyset$, or equivalently, there exist $B \in \beta$ such that $x \notin B$ holds for any $x \in U$. However, the specialization preordering \preceq_{β^c} can be still defined, and we have the following proposition characterizing the relationship between \preceq_β and \preceq_{β^c} .

Proposition 3 ([6]). For any covering β of U , the specialization preordering \preceq_{β^c} is precisely the dual of \preceq_β , namely $\preceq_{\beta^c} = \succeq_\beta$.

Definition 4 ([6]). For any covering β of U , $x \in U$, the point closure of x with respect to the subset system β^c is called the core of x , denoted by $\uparrow x$.

For any $x \in U$, $\uparrow x = \bigcap \{D \mid x \in D \in \beta^c\}$ and $\mathcal{P}(\beta^c) = \{\uparrow x \mid x \in U\}$. It is easy to verify that $\uparrow x$ is precisely the principal up-set $\{y \in U \mid x \preceq_\beta y\}$ generated by x with respect to \preceq_β . Note that $\mathcal{P}(\beta^c)$ is also a covering of U since $x \in \uparrow x$ holds for any $x \in U$.

Any covering β can also induces a binary relation T_β determined by $\bigcup \{B \times B \mid B \in \beta\}$. Moreover, $\forall x \in U$, $T_\beta(x) = \text{star}(x, \beta)$. Then we have the following relational characterizations of the three induced coverings.

Theorem 1. *For any covering β , point closure systems $\mathcal{P}(\beta) = \{\succeq_\beta(x) \mid x \in U\}$, $\mathcal{P}(\beta^c) = \{\preceq_\beta(x) \mid x \in U\}$, and star system $\mathcal{S}(\beta) = \{T_\beta(x) \mid x \in U\}$.*

2.2 Transformations among Induced Granular Worlds

Note that, according to Proposition 1, \mathcal{P} transforms a coarser granular world into finer one, while \mathcal{S} transforms a finer granular world into coarser one, so they can be understood as unary operations on covering simulating the process of switching among different granular worlds. Proposition 2 only tells us that $\mathcal{P}(\mathcal{P}(\beta)) = \mathcal{P}(\beta)$, namely \mathcal{P} is an idempotent operation. To completely understand these transformations, we need to study \mathcal{P} and \mathcal{S} further.

For any covering β , T_β is a tolerance, a binary relation satisfying reflexivity and symmetry [1]. Theorem 1 implies that tolerance relation will be an useful tool to investigate properties of \mathcal{S} , so we will firstly recall some basic definitions and facts about tolerance relation.

Let T be any tolerance relation on U , $x \in U$, the T -relative set $T(x)$ is always called T -class, and we will denote the family of all T -classes as $\mathfrak{C}(T)$, namely $\mathfrak{C}(T) = \{T(x) \mid x \in U\}$. Obviously, $\mathfrak{C}(T)$ is an covering induced by T , and for any covering β , $\mathcal{S}(\beta) = \mathfrak{C}(T_\beta)$.

A T -preblock is any subset $B \subseteq U$ such that xTy holds for any $x, y \in B$. Any subset $B \subseteq U$ is called a T -block if and only if it is a T -preblock such that for any $x \notin B$, there exists $b \in B$ satisfying $\neg(bTx)$. In the current paper, the family of all T -blocks will be denoted as $\mathfrak{B}(T)$, which is also an induced covering by T . It is well known that there exists one-to-one correspondence between the collection of all tolerance relations and the collection of all T -blocks on U . Moreover, we have the following Proposition 4, which not only characterizes the relationship between T -classes and T -blocks, but also implies that $\mathfrak{C}(T) = \mathcal{S}(\mathfrak{B}(T))$.

Proposition 4 ([1]). *Let (U, T) be any tolerance space, $x \in U$, $B \in \mathfrak{B}(T)$, then $T(x) = \bigcup \{B \mid x \in B\}$, $B = \bigcap \{T(x) \mid x \in B\}$.*

For any tolerance space (U, T) and $x \in U$, the intersection of T -classes containing x was called compatibility kernel $\langle x \rangle_T$ in [5], namely $\langle x \rangle_T = \bigcap \{T(y) \mid x \in T(y)\}$, and the family of all compatibility kernels was denoted by $\langle U \rangle_T$. Obviously, $\langle x \rangle_T = \downarrow x$, in which $\downarrow x$ is the point closure of x with respect to covering $\mathfrak{C}(T)$. Furthermore, we have $\langle U \rangle_T = \mathcal{P}(\mathfrak{C}(T))$. It was also proved in [5] that $\langle x \rangle_T = \bigcap \{B \mid x \in B \in \mathfrak{B}(T)\}$, or equivalently, $\mathcal{P}(\mathfrak{B}(T)) = \mathcal{P}(\mathfrak{C}(T))$. So in the following, the point closure system $\mathcal{P}(T)$ of a tolerance relation T is always referred to $\mathcal{P}(\mathfrak{B}(T))$ or $\mathcal{P}(\mathfrak{C}(T))$.

Proposition 5. Let (U, T) be any tolerance space, $x, y \in U$, $B \in \mathfrak{B}(T)$, then

1. $B = \bigcup \{\downarrow y \mid y \in B\}$, $T(x) = \bigcup \{\downarrow y \mid y \in T(x)\}$.
2. $\bigcup \{\downarrow y \mid x \in \downarrow y\} \subseteq T(x)$.

The proofs of the above proposition is trivial, however, it should be pointed out that $\bigcup \{\downarrow y \mid x \in \downarrow y\} \supseteq T(x)$ does not hold in general. Now we turn to discuss the relationship between $\mathcal{P}(\mathcal{S}(\beta))$ and $\mathcal{P}(\beta)$.

Proposition 6. For any covering β of U , $\mathcal{P}(\mathcal{S}(\beta)) = \mathcal{P}(T_\beta) \subseteq \mathcal{P}(\beta)$. Furthermore, if β is precisely $\mathfrak{B}(T)$ with respect to tolerance T , then $\mathcal{P}(\mathcal{S}(\beta)) = \mathcal{P}(\beta)$.

Proof. It is sufficient to prove that $\downarrow x \subseteq \langle x \rangle_{T_\beta}$ holds for any $x \in U$. Note that $y \in \downarrow x \Leftrightarrow \forall B \in \beta (x \in B \rightarrow y \in B)$, then if $x \in T_\beta(z) = \bigcup \{B \mid z \in B \in \beta\}$, there must exists $B_0 \in \beta$ such that $x \in B_0$ and $z \in B_0$. So $y \in B_0$ and $z \in B_0$ holds, which follows that $y \in T_\beta(z)$. Hence, $\forall z \in U (x \in T_\beta(z) \rightarrow y \in T_\beta(z))$, namely $\downarrow x \subseteq \langle x \rangle_{T_\beta}$.

If β is precisely $\mathfrak{B}(T)$ with respect to tolerance relation T , then $\downarrow x = \langle x \rangle_{T_\beta}$ holds for any $x \in U$.

Remark 1. The converse of proposition 6 is not true in general. Let $U = \{1, 2, 3\}$ and $\beta = \{\{1, 3\}, \{2, 3\}, \{3\}\}$, then $\downarrow x = \langle x \rangle$ holds for any $x \in U$, however, β is not the family of tolerance blocks at all.

For any tolerance relation T , $\mathfrak{C}(T)$, $\mathfrak{B}(T)$ and $\mathcal{P}(T)$ are all coverings of U induced by T , which represent granular worlds at different abstraction level. We also have known that $\mathcal{P}(\mathfrak{C}(T)) = \mathcal{P}(T) = \mathcal{P}(\mathfrak{B}(T))$ and $\mathcal{S}(\mathfrak{B}(T)) = \mathfrak{C}(T)$. In general, however, $T_{\mathcal{P}(T)}(x) \subseteq T(x)$ for any $x \in U$, or equivalently $\mathfrak{C}(T) \subseteq \mathcal{S}(\mathcal{P}(T))$, in which $T_{\mathcal{P}(T)}$ denotes the tolerance relation induced by $\mathcal{P}(T) = \{\langle x \rangle_T \mid x \in U\}$. The following theorem characterizes sufficient and necessary condition of $\mathcal{S}(\mathcal{P}(T)) = \mathfrak{C}(T)$, or equivalently $T = T_{\mathcal{P}(T)}$.

Theorem 2 ([5]). For any tolerance relation T , then $T_{\mathcal{P}(T)} = T$ if and only if, for any $a, b \in U$, aTb implies that there exists $c \in U$ such that $a \in \langle c \rangle_T$ and $b \in \langle c \rangle_T$.

An interesting conclusion of Proposition 6 and Theorem 2 is $\mathcal{S}(\mathcal{P}(\mathcal{S}(\mathcal{P}(\alpha)))) = \mathcal{S}(\mathcal{P}(\alpha))$ holds for any covering α , which implies that the composition of \mathcal{P} and \mathcal{S} is also idempotent.

Theorem 3. For any covering α of U , we denote $\mathcal{P}(\alpha)$ by β , then $T_{\mathcal{P}(T_\beta)} = T_\beta$.

Proof. Note that $\mathcal{P}(\mathcal{P}(\alpha)) = \mathcal{P}(\alpha)$ and $\beta = \mathcal{P}(\alpha)$, then $\downarrow_\alpha x = \downarrow_\beta x \triangleq \downarrow x$, in which $\downarrow_\alpha x$ and $\downarrow_\beta x$ denote the point closure of x with respect to α and β , respectively. By the proof of Proposition 6, $\downarrow x \subseteq \langle x \rangle_{T_\beta}$ holds for any $x \in U$.

For any $a, b \in U$, since $T_\beta(x) = \bigcup \{\downarrow y \mid x \in \downarrow y\}$, if $aT_\beta b$ then there exists $c \in U$ such that $a \in \downarrow c$ and $b \in \downarrow c$, so $a \in \langle c \rangle_{T_\beta}$ and $b \in \langle c \rangle_{T_\beta}$. According to Theorem 2, we have $T_{\mathcal{P}(T_\beta)} = T_\beta$.

3 Applications to Covering Rough Approximation

This section mainly discusses some applications of our multi-granular perspectives on covering. we firstly recall relationship between quasi-discrete closure operator and binary relation, and then discuss relational interpretations as well as axiomizations of four types of covering rough approximation operators.

3.1 Čech Closure Operators

Closure operators in the sense of Čech [3][7] generalize topological ones in that idempotent axiom does not necessarily holds.

Definition 5 ([3][7]). *Let $Cl : \wp(U) \rightarrow \wp(U)$ be any mapping, $\forall X, Y \subseteq U$, Cl is called Čech closure operator on U if it satisfies following axioms (C1)-(C3):*

- (C1) $Cl(\emptyset) = \emptyset$
- (C2) $X \subseteq Cl(X)$
- (C3) $Cl(X \cup Y) = Cl(X) \cup Cl(Y)$

If, in addition, Cl also satisfies quasi-discreteness axiom (C4), then we call it Čech quasi-discrete closure operator:

$$(C4) \quad Cl(X) = \bigcup \{Cl(x) \mid x \in X\}$$

in which $Cl(x)$ denotes the closure of singleton $\{x\}$.

Note that (C4) is equivalent to the following axiom (C4'):

$$(C4') \quad Cl(\bigcup \{X_i \mid i \in I\}) = \bigcup \{Cl(X_i) \mid i \in I\}$$

where $X_i \subseteq U$ for any $i \in I$. Any Čech closure operator Cl satisfying Kurotowski axiom (C5):

$$(C5) \quad Cl(X) = Cl(Cl(X))$$

is called topological closure operator, and any quasi-discrete topological closure operator is called Alexandroff closure operator.

Let Cl be any Čech closure operator on U , $\forall X \subseteq U$, the interior of X is defined as the complement of the closure of the complement of X . Formally, $Int(X) = -(Cl(-X))$, and we always call Int the interior operator. For any $x \in U$, if $x \in Int(X)$ then X is called a neighbourhood of x .

Proposition 7. *Let Cl be any Čech closure operator on U , $X, X_i \subseteq U$ and $x \in U$,*

1. $Cl(X) = \bigcup \{Cl(x) \mid x \in U\}$ *if and only if each element u of U has a minimal neighbourhood $N(u)$.*
2. $Cl(\bigcup \{X_i \mid i \in I\}) = \bigcup \{Cl(X_i) \mid i \in I\}$ *if and only if $Int(\bigcap \{X_i \mid i \in I\}) = \bigcap \{Int(X_i) \mid i \in I\}$.*

Moreover, Čech quasi-discrete closure operators are closely connected with binary relations on U .

Theorem 4 ([7]). *Let $Cl : \wp(U) \rightarrow \wp(U)$, then Cl is Čech quasi-discrete closure operator if and only if there exists binary relation $R \subseteq U \times U$ such that, for any $X \subseteq U$, $Cl(X) = Cl_R(X) = X \cup \{x \in U \mid R(x) \cap X \neq \emptyset\}$.*

For any relation R on U , let Int_R denote the dual of Cl_R , then $Int_R(X) = -Cl_R(-X) = \{x \in X \mid R(x) \subseteq X\}$, $N(u) = \{u\} \cup R(u)$, $Cl_R(u) = \{u\} \cup R^{-1}(u)$. By Theorem 4, if R is reflexive, $Cl_R(X) = \{x \in U \mid R(x) \cap X \neq \emptyset\}$. In fact, any reflexive relation on U bijectively corresponds to Čech quasi-discrete closure operator [3][7], which generates the well-known 1-1 correspondence between pre-ordering relations and Alexandroff topologies.

Proposition 8. *For any relation R on U , R is symmetric if and only if for any $u \in U$, $Cl_R(u)$ is the minimal neighbourhood $N(u)$ of u .*

3.2 Covering based Rough Approximation Operators

We first recall a type of generalized rough approximation operators based on arbitrary binary relation.

Definition 6 ([16]). *Let R be any binary relation on U , operators $\overline{R}, \underline{R} : \wp(U) \rightarrow \wp(U)$ are defined as follows: $\forall X \subseteq U$,*

$$\overline{R}(X) = \{x \in U \mid R(x) \cap X \neq \emptyset\}$$

$$\underline{R}(X) = \{x \in U \mid R(x) \subseteq X\}$$

We call $\overline{R}, \underline{R}$ generalized rough upper and lower approximation operators based on relation R .

Obviously, for any binary relation R on U , $X \subseteq U$, $Cl_R(X) = X \cup \overline{R}(X)$ (or equivalently, $Int_R(X) = X \cap \underline{R}(X)$). Particularly, if R is reflexive then $Cl_R(X) = \overline{R}(X)$ (or equivalently, $Int_R(X) = \underline{R}(X)$).

For any given covering space, different types of covering based rough approximation operators have been proposed [2][13][21][23]. In this subsection, we will mainly discuss relational interpretations and axiomatic characterizations of four types of covering based rough approximation operators.

Definition 7 ([22][20]). *For any covering β of U , operators $FH, FL : \wp(U) \rightarrow \wp(U)$ are defined as follows: $\forall X \subseteq U$,*

$$FH(X) = \bigcup \{B \in \beta \mid B \cap X \neq \emptyset\}$$

$$FL(X) = \{x \in U \mid \forall B \in \beta (x \in B \rightarrow B \subseteq X)\}$$

We call FH, FL the first type of covering upper and lower operators, respectively.

It is obvious that FH and FL are dual. Moreover, we have following relational interpretation of FH and FL .

Proposition 9. *For any covering β of U , let T_β denote the induced tolerance relation, then $\forall X \subseteq U$,*

$$FH(X) = \{x \in U \mid T_\beta(x) \cap X \neq \emptyset\}$$

$$FL(X) = \{x \in U \mid T_\beta(x) \subseteq X\}$$

Proof. Since FH and FL are dual operators, it is sufficient to prove only the half. $\forall y \in \bigcup \{B \in \beta \mid B \cap X \neq \emptyset\}$, there exists $B_0 \in \beta$ and $y \in B_0$, which follows $T_\beta(y) \cap X \neq \emptyset$, then $y \in \{x \in U \mid T_\beta(x) \cap X \neq \emptyset\}$. On the other hand, $\forall y \in \{x \in U \mid T_\beta(x) \cap X \neq \emptyset\}$, there exists $B_0 \in \beta$ such that $y \in B_0$ and $B_0 \cap X \neq \emptyset$, then $y \in \bigcup \{B \in \beta \mid B \cap X \neq \emptyset\}$.

Proposition 10. Let T_β be the induced tolerance relation of covering β , $\forall X \subseteq U$, then $FH(X) = Cl_\beta(X)$ and $FL(X) = Int_\beta(X)$, in which Cl_β and Int_β are Čech quasi-discrete closure and interior operators bijectively corresponding to tolerance relation T_β , respectively.

Proposition 10 implies that FH and FL are generalized rough upper and lower approximation operators based on tolerance relation T_β , or equivalently, granules $T_\beta(x) = star(x, \beta)$ of induced granular world $\mathcal{S}(\beta)$ are exploited as basic knowledge blocks to approximate uncertain knowledge. Furthermore, we have the following axiomatic characterization of FH .

Theorem 5. For any mapping $H : \wp(U) \rightarrow \wp(U)$, $\forall x, y \in U$, $X \subseteq U$ and $X_i \subseteq U$ ($\forall i \in I$), then H satisfies the following properties (1H)-(4H):

- (1H) $H(\emptyset) = \emptyset$
- (2H) $X \subseteq H(X)$
- (3H) $H(\bigcup X_i) = \bigcup H(X_i)$
- (4H) $y \in H(x) \Leftrightarrow x \in H(y)$

if and only if there exists a covering β such that $H = FH$.

Proof. Suppose H satisfies properties (1H)-(4H). We define $T \subseteq U \times U$: $\forall x, y \in U$, $xTy \Leftrightarrow y \in H(x)$, then T is tolerance relation and $T(x) = H(x)$ holds for any $x \in U$. Let $\beta = \mathfrak{B}(T)$, then $FH = Cl_{T_\beta}$ and $FH(x) = \bigcup \{B \in \beta \mid x \in B\} = T(x) = H(x)$. Furthermore, considering the quasi-discreteness of Cl_{T_β} , $\forall X \subseteq U$, we have $FH(X) = \bigcup FH(x) = \bigcup H(x) = H(X)$, which follows $FH = H$.

By Proposition 10, $FH(X) = Cl_\beta(X)$ holds for any $X \subseteq U$, so the proof of the sufficiency is trivial.

Remark 2. In the proof of sufficiency, the covering β is not unique. For example, the family of binary subsets $\{x, y\}$ such that $y \in H(x)$ is chosen as the covering β in [20].

The second type of covering rough approximation operators can be regarded as a special case of the first ones in the sense that the covering is an induced point closure system, however, their axiomatic characterization are different.

Definition 8 ([21][20]). For any covering β , operators $SH, SL : \wp(U) \rightarrow \wp(U)$ are defined as follows: $\forall X \subseteq U$,

$$SH(X) = \bigcup \{\downarrow x \mid \downarrow x \cap X \neq \emptyset\}$$

$$SL(X) = \{x \in U \mid \forall y \in U (x \in \downarrow y \rightarrow \downarrow y \subseteq X)\}$$

We call SH, SL the second type of covering upper and lower operators, respectively.

Then, in a completely similar way, we have the following Proposition 11 and Proposition 12.

Proposition 11. For any covering β of U , let $T_{\mathcal{P}(\beta)}$ denote the induced tolerance relation, then $\forall X \subseteq U$,

$$SH(X) = \{x \in U \mid T_{\mathcal{P}(\beta)}(x) \cap X \neq \emptyset\}$$

$$SL(X) = \{x \in U \mid T_{\mathcal{P}(\beta)}(x) \subseteq X\}$$

Proposition 12. For any $X \subseteq U$, $SH(X) = Cl_{\mathcal{P}(\beta)}(X)$ and $SL(X) = Int_{\mathcal{P}(\beta)}(X)$, in which $Cl_{\mathcal{P}(\beta)}$ and $Int_{\mathcal{P}(\beta)}$ are Cech quasi-discrete closure and interior operators bijectively corresponding to $T_{\mathcal{P}(\beta)}$, respectively.

Proposition 12 implies that SH and SL are generalized rough upper/lower approximation operators based on tolerance relation $T_{\mathcal{P}(\beta)}$, or equivalently, granules $T_{\mathcal{P}(\beta)}(x)$ of induced granular world $\mathcal{S}(\mathcal{P}(\beta))$ are basic knowledge blocks.

Theorem 6. Let $H : \wp(U) \rightarrow \wp(U)$, $\forall x, y \in U$, $X \subseteq U$ and $X_i \subseteq U$ holds for any i of arbitrary index set I , then there exists a covering β such that $H = SH$ if and only if H satisfies (1H)-(4H) and the following property (5H):

(5H) if $y \in H(x)$, there exists $u \in U$ such that $x, y \in \bigcap \{H(z) \mid u \in H(z)\}$

Proof. Suppose H satisfies properties (1H)-(4H). We define $T \subseteq U \times U$: $\forall x, y \in U$, $xTy \Leftrightarrow y \in H(x)$, then T is tolerance relation and $T(x) = H(x)$ holds for any $x \in U$. Note that $\langle x \rangle_T = \bigcap \{H(z) \mid x \in H(z)\}$, so according to Theorem 2, we have $T = T_{\mathcal{P}(T)}$.

Let $\beta = \mathfrak{B}(T)$, then $T_{\mathcal{P}(\beta)} = T_{\mathcal{P}(T)} = T$, so for any $x \in U$, $SH(x) = \bigcup \{\downarrow y \mid x \in \downarrow y\} = T(x) = H(x)$. Considering the quasi-discreteness of $Cl_{\mathcal{P}(\beta)}$, we have $SH(X) = \bigcup \{SH(x) \mid x \in X\} = \bigcup \{H(x) \mid x \in X\} = H(X)$, which follows $SH = H$.

Suppose β be any covering of U , then $SH = Cl_{\mathcal{P}(\beta)}$. According to Theorem 5, SH satisfies axioms (1H)-(4H); Moreover, by theorem 2 and theorem 3, SH also satisfies axiom (5H).

Remark 3. It should be pointed out that the covering β is also not unique in the proof of sufficient condition. For example, the family of subsets $\{H(x) \mid x \in U\}$ is chosen as the covering β in [20].

The third type of covering based rough approximation operators on finite universe was defined in [22], however, the upper and the lower approximation operator were not dual. In the following, we will discuss the third type of covering based upper approximation operator and its dual, and our results also apply to infinite universe [11].

Definition 9 ([22]). For any covering β of U , operators $TH, TL : P(U) \rightarrow P(U)$ are defined as follows: $\forall X \subseteq U$,

$$\begin{aligned} TH(X) &= \bigcup \{\downarrow x \mid x \in X\} \\ TL(X) &= \{x \in U \mid \forall u(x \in \downarrow u \rightarrow \downarrow u \subseteq X)\} \end{aligned}$$

We call TH, TL the third type of covering upper and lower operators, respectively.

Proposition 13. For any covering β of U , $\forall X \subseteq U$,

$$\begin{aligned} TH(X) &= \{x \in U \mid \uparrow x \cap X \neq \emptyset\} \\ TL(X) &= \{x \in U \mid \uparrow x \subseteq X\} \end{aligned}$$

Proof. For any $X \subseteq U$, then $TH(X) = \bigcup \{\downarrow y \mid y \in X\} = \{x \in U \mid \exists y \in X(x \in \downarrow y)\} = \{x \in U \mid \uparrow x \cap X \neq \emptyset\}$. Dually, $TL(X) = \{x \in U \mid \uparrow x \subseteq X\}$.

According to Theorem 1, $\uparrow x = \preceq_\beta(x)$, then operators TH and TL are in fact Čech quasi-discrete closure and interior operators bijectively corresponding to the reflexive and transitive relation \preceq_β .

Proposition 14. *For any $X \subseteq U$, $TH(X) = Cl_{\preceq_\beta}(X)$, $TL(X) = Int_{\preceq_\beta}(X)$.*

Theorem 7. *Let $H : P(U) \rightarrow P(U)$, $X \subseteq U$, $X_i \subseteq U (\forall i \in I)$, then H satisfies the following properties (H1)-(H4):*

- (H1) $H(\emptyset) = \emptyset$
- (H2) $X \subseteq H(X)$
- (H3) $H(\bigcup X_i) = \bigcup H(X_i)$
- (H4) $H(H(X)) = H(X)$

if and only if there exists a covering β such that $H = TH$.

Proof. For any covering β , $TH = Cl_{\preceq_\beta}$, which follows that TH satisfies properties (H1)-(H4).

Suppose H satisfies properties (H1)-(H4) and $\forall x \in U$ By property (H2) then $x \in H(x)$ holds, which follows that $\beta = \{H(u) \mid u \in U\}$ is a covering of U . By property (H4), $H(H(x)) = H(x)$ and $\downarrow x = \bigcap \{H(y) \mid x \in H(y)\} = H(x)$, so $TH(x) = \{y \mid \uparrow y \cap \{x\} \neq \emptyset\} = \{y \mid x \in \uparrow y\} = \downarrow x$. For any $X \subseteq U$, by the quasi-discreteness of TH as well as the property (H3), then $TH(X) = \bigcup \{TH(x) \mid x \in X\} = \bigcup \{H(x) \mid x \in X\} = H(X)$.

The fourth type of covering based rough approximation operators was defined in [15][23], and [4][14] discussed a special case in that the covering is precisely $\mathfrak{B}(T)$ of given tolerance relation T . In [11][15], it was proved that the fourth type of operators are equivalent to rough approximation operators based on preordering relation.

Definition 10 ([23]). *For any covering β of U , operators $XH, XL : P(U) \rightarrow P(U)$ are defined as follows: $\forall X \subseteq U$,*

$$\begin{aligned} XH(X) &= \{x \in U \mid \downarrow x \cap X \neq \emptyset\} \\ XL(X) &= \{x \in U \mid \downarrow x \subseteq X\} \end{aligned}$$

We call XH, XL the fourth type of covering upper and lower operators, respectively.

According to Theorem 1, $\downarrow x = \succeq_\beta(x)$, so we have the following topological interpretation of XH and XL .

Proposition 15. *For any $X \subseteq U$, $XH(X) = Cl_{\succeq_\beta}(X)$, $XL(X) = Int_{\succeq_\beta}(X)$.*

Theorem 8. *Let $H : P(U) \rightarrow P(U)$, $\forall x, y \in U$, $X \subseteq U$ and $X_i \subseteq U$ holds for any i of arbitrary index set I , then H satisfies properties (H1)-(H4) if and only if there exists a covering β such that $H = XH$.*

Proof. Suppose H satisfies properties (H1)-(H4). Let $\beta = \{K^c \mid H(K) = K\}$, then β is a covering of U since $H(\emptyset) = \emptyset$, and $\forall x \in U$, it is easy to verify that the point closure of x with respect to β^c is precisely $H(x)$, and that $XH(x) = \uparrow x$, the core of x with respect to β , so $XH(x) = H(x)$. For any $X \subseteq U$, by the quasi-discreteness of XH and property (H3), we have $XH(X) = \bigcup \{XH(x) \mid x \in X\} = \bigcup \{H(x) \mid x \in X\} = H(X)$.

Note that the third and fourth types of upper approximation operators have the same axiomatic characterization, however, the corresponding coverings are different.

4 Conclusion

Any covering can be associated with at least two new ones: point closure system and star system, both of which have obvious relational interpretation. The paper understands these coverings as granular worlds at different abstraction level, and investigates typical properties of transformations among these granular worlds.

As an application of the presented multi-granular perspectives on covering, four types of covering based rough approximation operators are proved to be induced by binary relations, which, to some extent, bridges the gap between generalized rough sets based on binary relation and coverings. Furthermore, axiomatic characterizations of these four types of upper approximation operators are obtained, and these results can be dually apply to lower ones.

References

1. Bartol, W., Miró, J., Pióro, K., Rosselló, F.: On the Coverings by Tolerance Classes. *Information Science*, 166, 193–211 (2004)
2. Bonikowski, Z., Bryniarski, E., Wybraniec-Skardowska, U.: Extensions and Intentions in the Rough Set Theory. *Information Science*, 107, 149–167 (1998)
3. Čech, E.: *Topological Spaces*. Wiley, New York (1966)
4. Chen, W. L., Cheng, J. X., Zhang, C. J.: Generalized Rough Set Model Based on Compatibility Relation. *Journal of Computer Engineering and Application* (in Chinese), 40(4), 26–28 (2004)
5. Cheng, J. X., Chen, W. L., Song, J.: Compatibility Relation and Compatibility Kernel. *Journal of Anhui University* (in Chinese), 28(4), 1–4 (2004)
6. Ern , M.: The ABC of Order and Topology. In: H. Herrlich and H. E. Porst (eds.), *Category Theory at Work*. Heldermann, Berlin, pp. 57–83 (1991)
7. Galton, A.A.: A Generalized Topological View of Motion in Discrete Space. *Theoretical Computer Science*, 305, 111–134 (2003)
8. Ganter, B., Wille, R.: *Formal Concept Analysis – Mathematical Foundations*. Springer (1999)
9. Hobbs, J. R.: Granularity. In: *Proceedings of the 9th International Joint Conference on Artificial Intelligence*, pp. 432–435 (1985)
10. Lin, T. Y.: Granular Computing I: the Concept of Granulation and Its Formal Model. *Int. J. of Granular Computing, Rough Sets and Intelligent Systems*, 1(1), 21–42 (2009)
11. Liu, G. L.: A Comparison of Two Types of Rough Sets Induced by Covering. *Int. J. of Approximate Reasoning*, 50, 521–528 (2009)
12. Pawlak, Z.: *Rough Sets: Theoretical Aspects of Reasoning about Data*. Kluwer Academic Publishers (1991)
13. Qin, K., Zeng, Z.: On Covering Rough Sets. *LNCS*, vol 4481, pp. 34–41 (2007)
14. Wu, C., Hu, X. H., Yang, J. Y., X. B. Yang, X. B.: Expanding Tolerance RST Models Based Cores of Maximal Compatible Blocks. *LNAI*, vol 4259, pp. 235–243 (2006)

15. Xu, Z. Y., Wang, Q.: Properties of Covering Rough Set Model. *Journal of Henan Normal University* (in Chinese), 33(1), 130–132 (2005)
16. Yao, Y. Y.: Relational Interpretations of Neighborhood Operators and Rough Set Approximation Operators. *Information Science*, 111, 239–259 (1998)
17. Zadeh, L. A.: Fuzzy Logic = Computing with Words. *IEEE Transaction on Fuzzy Systems*, 4(2), 103–111 (1996)
18. Zadeh, L. A.: Towards a Theory of Fuzzy Information Granulation and Its Centrality in Human Reasoning and Fuzzy Logic. *Fuzzy Sets and Systems*, 90, 111–127 (1997)
19. Zhang, L., Zhang, B.: *Theory and Application of Problem Solving*. Elsevier, North-Holland (1992)
20. Zhang, Y. L., Li, J. J., Wu, W. Z.: On Axiomatic Characterizations of Three Pairs of Covering Based Approximate Operators. *Information Science*, 180, 274–287 (2010)
21. Zhu, W., F. Y. Wang, F.W.: Reduction and Axiomization of Covering Generalized Rough Sets. *Information Sciences*, 152, 217–230 (2003)
22. Zhu, W.: Topological Approaches to Covering Rough Sets. *Information Science*, 177, 1499–1508 (2007)
23. Zhu, W.: Relationship between Generalized Rough Sets based on Binary Relation and Covering. *Information Science*, 179, 210–225 (2009)